Math 228
3-2-1994

Solutions to Problem Set 18.

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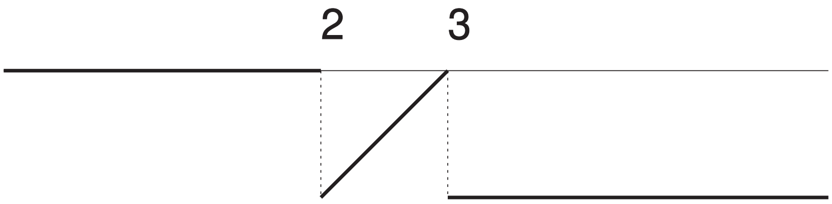
2.Sketch the graph of $(t-3)u_2(t) - (t-2)u_3(t)$ on the interval $t \geq 0$.



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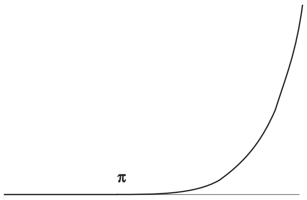
3. Sketch the graph of $f(t-\pi)u_\pi(t)$, where $f(t) = t^2$, on the interval $t \geq 0$.



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8. Find the Laplace transform of the function

$$f(t) = \begin{cases} 0 & \text{if } t < 1 \\ t^2 - 2t + 2 & \text{if } t \geq 1 \end{cases}$$

First, write f in terms of step functions:

$$f(t) = (t^2 - 2t + 2)u_1(t).$$

Next, match $(t^2 - 2t + 2)u_1(t)$ with the left side of the transform formula

$$h(t-a)u_a(t) \quad \rightarrow \quad e^{-as}H(s)$$

(I'm using h and H because f was already used for the name of the given function.)

$u_a(t) = u_1(t)$, so $a = 1$. Likewise, $h(t-a) = h(t-1) = t^2 - 2t + 2$. To apply the formula, I need $H(s)$. This is the transform of $h(t)$, *but what I have is* $h(t-1)$. To get $h(t)$ from $h(t-1)$, replace t with $t+1$:

$$h\left((t+1) - 1\right) = (t+1)^2 - 2(t+1) + 2,$$

$$h(t) = t^2 + 1.$$

Since $\mathcal{L}(t^2) = \dfrac{2}{s^3}$ and $\mathcal{L}(1) = \dfrac{1}{s}$,

$$H(s) = \mathcal{L}\left(h(t)\right) = \mathcal{L}(t^2 + 1) = \frac{2}{s^3} + \frac{1}{s}.$$

Therefore,

$$\mathcal{L}\left(f(t)\right) = \mathcal{L}\left((t^2 - 2t + 2)u_1(t)\right) = e^{-s} \cdot \left(\frac{2}{s^3} + \frac{1}{s}\right). \ \square$$

9\. Find the Laplace transform of the function

$$f(t) = \begin{cases} 0 & \text{if } t < \pi \\ t - \pi & \text{if } \pi \le t < 2\pi \\ 0 & \text{if } t \ge 2\pi \end{cases}$$

$$f(t) = (t - \pi)(u_\pi(t) - u_{2\pi}(t)).$$

The transform is

$$\mathcal{L}\left((t - \pi)(u_\pi(t) - u_{2\pi}(t))\right) = \mathcal{L}\left((t - \pi)u_\pi(t)\right) - \mathcal{L}\left((t - 2\pi)u_{2\pi}(t)\right) - \pi\,\mathcal{L}\left(u_{2\pi}(t)\right) =$$

$$e^{-\pi s} \cdot \frac{1}{s^2} - e^{-2\pi s} \cdot \frac{1}{s^2} - \pi e^{-2\pi s} \cdot \frac{1}{s}. \ \square$$

10\. Find the Laplace transform of the function $f(t) = u_1(t) + 2u_3(t) - 6u_4(t)$.

$$\mathcal{L}\left(u_1(t) + 2u_3(t) - 6u_4(t)\right) = \left(e^{-s} + 2e^{-3s} - 6e^{-4s}\right) \cdot \frac{1}{s}. \ \square$$

15\. Find the inverse Laplace transform of the function $F(s) = \dfrac{2(s-1)e^{-2s}}{s^2 - 2s + 2}$.

This is a step function shift (e^{-2s}) *and* an exponential shift ($s^2 - 2s + 2 = (s-1)^2 + 1$). First, I'll invert the non-e^{-2s} part of F. Complete the square:

$$\mathcal{L}^{-1}\left(\frac{2(s-1)}{s^2 - 2s + 2}\right) = \mathcal{L}^{-1}\left(\frac{2(s-1)}{(s-1)^2 + 1}\right).$$

To invert this, I use an exponential shift — $e^{ct}f(t) \to F(s - c)$. In this case, $c = 1$ and $F(s) = \dfrac{2s}{s^2 + 1}$. Check:

$$F(s - 1) = \frac{2(s-1)}{(s-1)^2 + 1}.$$

Now $f(t)$ is the inverse transform of $F(s) = \dfrac{2s}{s^2 + 1}$, which is $2\cos t$. So applying the exponential shift,

$$\mathcal{L}^{-1}\left(\frac{2(s-1)}{(s-1)^2 + 1}\right) = 2e^t \cos t.$$

Now go back to the original problem. To apply the step function shift, replace the t's with $(t - 2)$'s (because of the factor e^{-2s}) and tack on $u_2(t)$:

$$\mathcal{L}^{-1}\left(\frac{2(s-1)e^{-2s}}{s^2 - 2s + 2}\right) = 2e^{t-2}\cos(t - 2)u_2(t). \ \square$$

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16. Find the inverse Laplace transform of the function $F(s) = \dfrac{2e^{-2s}}{s^2 - 4}$.

$$\mathcal{L}^{-1}\left(\frac{2}{s^2 - 4}\right) = \sinh 2t,$$

so

$$\mathcal{L}^{-1}\left(\frac{2e^{-2s}}{s^2 - 4}\right) = \sinh 2(t - 2) \cdot u_2(t). \square$$

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28. Let f satisfy $f(t + T) = f(t)$ for all $t \geq 0$ and for some fixed positive number T; f is said to be periodic with period T on $0 \leq t < \infty$. Show that

$$\mathcal{L}(f(t)) = \frac{\int_0^T e^{-st} f(t)\,dt}{1 - e^{-sT}}.$$

The idea is to break the interval $0 \leq t < \infty$ up into intervals

$$0 \leq t \leq T, \quad T \leq t \leq 2T, \quad 2T \leq t \leq 3T, \ldots$$

On each interval, f is the same as it is on $0 \leq t \leq T$ by periodicity.

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)\,dt = \sum_{n=0}^\infty \int_{nT}^{(n+1)T} e^{-st} f(t)\,dt.$$

Substitute $u = t - nT$. Then $du = dt$, and the limits $t = nT$ and $t = (n+1)T$ become $u = 0$ and $u = T$, respectively. Also,

$$e^{-st} = e^{-s(u+nT)} = e^{-snT} e^{-su}, \quad f(t) = f(u + nT) = f(u).$$

($f(u + nT) = f(u)$ is true because f repeats every T units.)
Therefore,

$$\sum_{n=0}^\infty \int_0^T e^{-snT} e^{-su} f(u)\,du = \left[\sum_{n=0}^\infty \left(e^{-sT}\right)^n\right] \int_0^T e^{-su} f(u)\,du.$$

The series is geometric with ratio e^{-sT}, so its sum is $\dfrac{1}{1 - e^{-sT}}$. Therefore,

$$\mathcal{L}(f(t)) = \frac{\int_0^T e^{-su} f(u)\,du}{1 - e^{-sT}}. \square$$

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29. Use the result of Problem 28 to find the Laplace transform of

$$f(t) = \begin{cases} 1 & \text{if } 0 \leq t < 1 \\ 0 & \text{if } 1 \leq t < 2 \end{cases} \qquad f(t + 2) = f(t).$$

In this case, $T = 2$, so

$$\mathcal{L}(f(t)) = \frac{\int_0^2 e^{-st} f(t)\,dt}{1 - e^{-2s}} = \frac{\int_0^1 e^{-st}\,dt}{1 - e^{-2s}} = \frac{1 - e^{-s}}{s(1 - e^{-2s})}. \square$$

"Man is ready to die for an idea, provided that idea is not quite clear to him." - PAUL ELDRIDGE

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