Solutions to Problem Set 19.

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3. Find the solution of the initial value problem $y'' + 4y = \sin t - u_{2\pi}(t)\sin(t - 2\pi)$, $y(0) = 0$, $y'(0) = 0$.

Transform the given equation:

$$\left(0 - 0 + s^2 Y(s)\right) + 4Y(s) = \frac{1}{s^2+1} - e^{-2\pi s}\frac{1}{s^2+1}.$$

Solve for $Y(s)$:

$$Y(s) = \frac{1}{(s^2+1)(s^2+4)} - e^{-2\pi s}\frac{1}{(s^2+1)(s^2+4)}.$$

By partial fractions (or inspection),

$$\frac{1}{(s^2+1)(s^2+4)} = \frac{1}{3}\left(\frac{1}{s^2+1} - \frac{1}{s^2+4}\right).$$

Therefore,

$$\mathcal{L}^{-1}\left(\frac{1}{(s^2+1)(s^2+4)}\right) = \frac{1}{2}\left(\sin t - \frac{1}{2}\sin 2t\right).$$

The solution is

$$y(t) = \frac{1}{3}\sin t - \frac{1}{6}\sin 2t - u_{2\pi}(t)\left(\frac{1}{3}\sin(t - 2\pi) - \frac{1}{6}\sin 2(t - 2\pi)\right). \square$$

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8. Find the solution of the initial value problem $y'' + y' + \frac{5}{4}y = t - u_{\pi/2}(t)(\left(t - \frac{5}{4}\right)$, $y(0) = 0$, $y'(0) = 0$.

Transform the given equation, using the step function formula on the right:

$$s^2Y(s) + sY(s) + \frac{5}{4}Y(s) = \frac{1}{s^2} - \frac{1}{s^2}e^{-\pi s/2}.$$

Solve for $Y(s)$:

$$Y(s) = \frac{1}{s^2\left(s^2 + s + \frac{5}{4}\right)} - e^{-\pi s/2}\frac{1}{s^2\left(s^2 + s + \frac{5}{4}\right)}.$$

This is going to get rather technical, so let me explain what I'm about to do.
Let

$$G(s) = \frac{1}{s^2\left(s^2 + s + \frac{5}{4}\right)}.$$

Suppose I knew the inverse transform $\mathcal{L}^{-1}(G(s)) = g(t)$. Then the step function inverse transform gives the solution

$$y(t) = g(t) - u_{\pi/2}(t)g\left(t - \frac{\pi}{2}\right).$$

The work that follows will be devoted to computing $g(t)$. If you're willing to take the partial fractions for granted, skip ahead till you reach a "Yow!".

First, use partial fractions to decompose $\dfrac{1}{s^2\left(s^2+s+\dfrac{5}{4}\right)}$:

$$\frac{1}{s^2\left(s^2+s+\dfrac{5}{4}\right)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+s+\dfrac{5}{4}},$$

$$1 = As(s^2+s+\frac{5}{4}) + B(s^2+s+\frac{5}{4}) + (Cs+D)s^2.$$

Setting $s=0$ yields $1 = \frac{5}{4}$, so $B = \frac{4}{5}$. So

$$1 = As(s^2+s+\frac{5}{4}) + \frac{4}{5}(s^2+s+\frac{5}{4}) + (Cs+D)s^2,$$

$$1 = A\left(s^3+s^2+\frac{5}{4}s\right) + \frac{4}{5}(s^2+s+\frac{5}{4}) + Cs^3 + Ds^2.$$

Differentiate:

$$0 = A\left(3s^2+2s+\frac{5}{4}\right) + \frac{4}{5}(2s+1) + 3Cs^2 + 2Ds.$$

Setting $s=0$ yields $0 = \frac{5}{4}A + \frac{4}{5}$, so $A = -\frac{16}{25}$. Plug it back in:

$$0 = -\frac{16}{25}\left(3s^2+2s+\frac{5}{4}\right) + \frac{4}{5}(2s+1) + 3Cs^2 + 2Ds.$$

Differentiate again:

$$0 = -\frac{16}{25}(6s+2) + \frac{8}{5} + 6Cs + 2D.$$

Let $s=0$. Then $0 = -\frac{32}{25} + \frac{8}{5} + 2D$, so $D = -\frac{4}{25}$. Plug it back in:

$$0 = -\frac{16}{25}(6s+2) + \frac{8}{5} + 6Cs - \frac{8}{25}.$$

Differentiate one last time:

$$0 = -\frac{96}{25} + 6C \quad \text{so} \quad C = \frac{16}{25}.$$

Therefore,

$$\frac{1}{s^2\left(s^2+s+\dfrac{5}{4}\right)} = -\frac{16}{25}\frac{1}{s} + \frac{4}{5}\frac{1}{s^2} + \frac{\dfrac{16}{25}s - \dfrac{4}{25}}{s^2+s+\dfrac{5}{4}}.$$

(Yow!)

The first two terms can be inverted using a table. The last term is a quadratic with a middle term. I must complete the square, then use the exponential shift.

Observe first that

$$s^2+s+\frac{5}{4} = \left(s+\frac{1}{2}\right)^2 + 1.$$

This means that I want to express everything in terms of $s+\frac{1}{2}$. So

$$\frac{\frac{16}{25}s-\frac{4}{25}}{s^2+s+\frac{5}{4}}=\frac{\frac{16}{25}s-\frac{4}{25}}{\left(s+\frac{1}{2}\right)^2+1}=\frac{\frac{16}{25}\left(s+\frac{1}{2}\right)-\frac{12}{25}}{\left(s+\frac{1}{2}\right)^2+1}=\frac{16}{25}\frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+1}-\frac{12}{25}\frac{1}{\left(s+\frac{1}{2}\right)^2+1}.$$

(This kind of thing is enough to send anyone screaming into the night!)
Therefore,

$$G(s)=\frac{1}{s^2\left(s^2+s+\frac{5}{4}\right)}=-\frac{16}{25}\frac{1}{s}+\frac{4}{5}\frac{1}{s^2}+\frac{16}{25}\frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+1}-\frac{12}{25}\frac{1}{\left(s+\frac{1}{2}\right)^2+1}.$$

The inverse transform of this mess is

$$g(t)=-\frac{16}{25}+\frac{4}{5}t+\frac{16}{25}e^{-t/2}\cos t-\frac{12}{25}e^{-t/2}\sin t.$$

Remember (a long time ago!) that I was trying to find $g(t)$. Well there it is! Recall that

$$Y(s)=\frac{1}{s^2\left(s^2+s+\frac{5}{4}\right)}-e^{-\pi s/2}\frac{1}{s^2\left(s^2+s+\frac{5}{4}\right)}.$$

$g(t)$ is the inverse transform of $\dfrac{1}{s^2\left(s^2+s+\frac{5}{4}\right)}$. Hence, the first term inverts to $g(t)$. For the second term, shift g by $\frac{\pi}{2}$ and tack on the step function. The answer is

$$y(t)=g(t)-u_{\pi/2}(t)g\left(t-\frac{\pi}{2}\right).\ \Box$$

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9. Find the solution of the initial value problem $y''+y=g(t)$, $y(0)=1$, $y'(0)=1$, where

$$g(t)=\begin{cases} t & \text{if } 0\le t<1 \\ 1 & \text{if } t\ge 1\end{cases}$$

First,

$$g(t)=t\left(u_0(t)-u_1(t)\right)+u_1(t)=tu_0(t)-(t-1)u_1(t).$$

Insert this into the differential equation, and transform both sides:

$$\left(-1-0+s^2Y(s)\right)+Y(s)=\frac{1}{s^2}-\frac{1}{s^2}e^{-s}.$$

Solve for $Y(s)$:

$$Y(s)=\frac{1}{s^2+1}+\frac{1}{s^2(s^2+1)}-\frac{1}{s^2(s^2+1)}e^{-s}.$$

Notice that

$$\frac{1}{s^2(s^2+1)}=\frac{1}{s^2}-\frac{1}{s^2+1}.$$

Therefore,

$$\mathcal{L}^{-1}\left(\frac{1}{s^2(s^2+1)}\right) = t - \sin t.$$

The step function formula then gives

$$\mathcal{L}^{-1}\left(\frac{1}{s^2(s^2+1)}e^{-s}\right) = u_1(t)\left((t-1) - \sin(t-1)\right).$$

The solution is

$$y(t) = \sin t + t - \sin t - u_1(t)\left((t-1) - \sin(t-1)\right) = t - u_1(t)\left((t-1) - \sin(t-1)\right). \square$$

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11. Find the solution of the initial value problem $y'' + 4y = u_\pi(t) - u_{2\pi}(t)$, $y(0) = 1$, $y'(0) = 0$.

Transform the given equation:

$$\left(0 - s + s^2 Y(s)\right) + 4Y(s) = e^{-\pi s} \cdot \frac{1}{s} - e^{-2\pi s} \cdot \frac{1}{s}.$$

Solve for $Y(s)$:

$$Y(s) = \frac{s}{s^2+4} + e^{-\pi s} \cdot \frac{1}{s(s^2+4)} - e^{-2\pi s} \cdot \frac{1}{s(s^2+4)}.$$

By partial fractions,

$$\frac{1}{s(s^2+4)} = \frac{1}{4}\left(\frac{1}{s} - \frac{s}{s^2+4}\right).$$

Therefore,

$$\mathcal{L}^{-1}\left(\frac{1}{s(s^2+4)}\right) = \frac{1}{4}\mathcal{L}^{-1}\left(\frac{1}{s} - \frac{s}{s^2+4}\right) = \frac{1}{4}\left(1 - \cos 2t\right).$$

The solution is

$$y(t) = \cos 2t + \frac{1}{4}u_\pi(t)\left(1 - \cos 2(t-\pi)\right) - \frac{1}{4}u_{2\pi}(t)\left(1 - \cos 2(t-2\pi)\right). \square$$

"The great virtue in life is real courage, that knows how to face facts and live beyond them." - D. H. LAWRENCE