Math 228
1-20-1994

Solutions to Problem Set 1.

[page 9]

1. Determine the order of the differential equation $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+2y=\sin x$; also state whether the equation is linear or nonlinear.

$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+2y=\sin x$ is linear, and has order 2. $\Box$

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4. Determine the order of the differential equation $\frac{dy}{dx}+xy^2=0$; also state whether the equation is linear or nonlinear.

$\frac{dy}{dx}+xy^2=0$ is nonlinear, and has order 1. $\Box$

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11. Verify that $y_1=x^{1/2}$ and $y_2=x^{-1}$ are solutions to $2x^2y''+3xy'-y=0$, $x>0$.

Compute the derivatives:

$$y_1=x^{1/2},\quad y_1'=\frac{1}{2}x^{-1/2},\quad y_1''=-\frac{1}{4}x^{-3/2}.$$

Plug them into the equation:

$$2x^2y_1''+3xy_1'-y_1=(2x^2)(-\frac{1}{4}x^{-3/2})+(3x)(\frac{1}{2}x^{-1/2})-x^{1/2}=-\frac{1}{2}x^{1/2}+\frac{3}{2}x^{1/2}-x^{1/2}=0.$$

Compute the derivatives:

$$y_2=x^{-1},\quad y_2'=-x^{-2},\quad y_2''=2x^{-3}.$$

Plug them into the equation:

$$2x^2y_2''+3xy_2'-y_2=(2x^2)(2x^{-3})+(3x)(-x^{-2})-x^{-1}=4x^{-1}-3x^{-1}-x^{-1}=0.\ \Box$$

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17. Determine the values of r for which the differential equation $y''+y'-6y=0$ has solutions of the form $y=e^{rx}$.

Compute the derivatives:

$$y=e^{rx},\quad y'=re^{rx},\quad y''=r^2e^{rx}.$$

Plug them into the equation:

$$y''+y'-6y=r^2e^{rx}+re^{rx}-6e^{rx}=(r^2+r-6)e^{rx}=0.$$

Since this is supposed to be an identity, $r^2+r-6=0$, or $(r+3)(r-2)=0$. Therefore, $r=-3$ and $r=2$, so the solutions are e^{-3x} and e^{2x}. $\Box$

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19. Determine the values of r for which the differential equation $x^2y'' + 4xy' + 2y = 0$ has solutions of the form $y = x^r$ for $x > 0$.

Compute the derivatives:

$$y = x^r, \quad y' = rx^{r-1}, \quad y'' = r(r-1)x^{r-2}.$$

Plug them into the equation:

$$(x^2)(r)(r-1)(x^{r-2}) + (4x)(r)(x^{r-1}) + 2x^r = (r(r-1) + 4r + 2)\, x^2 = 0.$$

This equation is supposed to be an identity, so $r^2 + 3r + 2 = 0$, so $r = -2$ or $r = -1$. The solutions are $y = x^{-1}$ and $y = x^{-2}$.

The equation $x^2y'' + 4xy' + 2y = 0$ is an example of an **Euler equation**; see page 143. □

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24. Determine the order of the partial differential equation $u_{xx} + u_{yy} + uu_x + uu_y + u = 0$; also state whether the equation is linear or nonlinear.

The equation is nonlinear (because of the uu_x and uu_y terms) and has order 2 (because of the second derivatives u_{xx} and u_{yy}). □

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30. Verify that $u = (x^2 + y^2 + z^2)^{-1/2}$ for $(x, y, z) \neq (0, 0, 0)$ is a solution to the partial differential equation $u_{xx} + u_{yy} + u_{zz} = 0$.

First,

$$u_x = -x(x^2 + y^2 + z^2)^{-3/2} \quad \text{and} \quad u_{xx} = -(x^2 + y^2 + z^2)^{-3/2} + (3x^2)(x^2 + y^2 + z^2)^{-5/2}.$$

There's no need to compute u_{yy} and u_{zz} directly; since u is symmetric in x, y, and z, you can use symmetry. For example, to get u_{yy} from u_{xx}, just swap the x's and y's. Hence

$$u_{xx} + u_{yy} + u_{zz} = -3(x^2 + y^2 + z^2)^{-3/2} + (3x^2 + 3y^2 + 3z^2)(x^2 + y^2 + z^2)^{-5/2} =$$

$$-3(x^2 + y^2 + z^2)^{-3/2} + 3(x^2 + y^2 + z^2)^{-3/2} = 0. \square$$

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39. Sketch the direction field of the differential equation $y' = y(4-y)$. Based on the direction field, determine the behavior of y as $x \to \infty$.

You can use a computer to sketch the direction field, but it is easier than that.

$\dfrac{dy}{dx} = 0$ for $y = 0$ or $y = 4$. For these values, y is constant. $y =$ constant is a horizontal line, so the lines $y = 0$ and $y = 4$ are solutions.

$\dfrac{dy}{dx} = y(4 - y)$, so:

- $\dfrac{dy}{dx} > 0$ for $0 < y < 4$, so the solution curves *increase* for $0 < y < 4$.
- $\dfrac{dy}{dx} < 0$ for $y < 0$ or for $y > 4$, so the solution curves *decrease* for $y < 0$ or for $y > 4$.

Draw increasing lines between $y = 0$ and $y = 4$ and decreasing lines above $y = 4$ and below $y = 0$. Then "bend" the lines into curves so that the curves don't run into each other or the solutions $y = 0$ and $y = 4$.

(Reason: Under normal conditions, there is a unique solution curve passing through each point: Solution curves don't cross.)

Here's how to do it using *Mathematica*:

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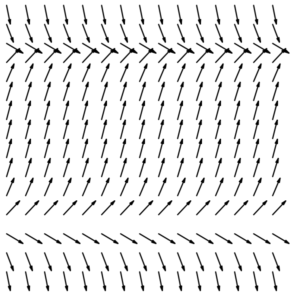
Needs["Graphics`PlotField`"]

PlotVectorField[{1, y(4 - y)}, {x, 0, 6}, {y, -1, 5},
    ScaleFunction->(1 &)]
```



As $x \to \infty$:

- $y \to 4$ for $y > 4$, for $0 < y < 4$, and obviously for $y = 4$.
- $y \to -\infty$ for $y < 0$.
- $y \to 0$ for $y = 0$. $\square$

"A young man who is not a radical about something is a pretty poor risk for education." - JACQUES BARZUN, *Teacher in America*