Math 228
3-25-1994

Solutions to Problem Set 23.

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15. Find all eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix}$$

The characteristic polynomial is $8 - 6\lambda + \lambda^2$. The eigenvalues are $\lambda = 4$ and $\lambda = 2$.

Consider $\lambda = 4$. To find an eigenvector $\vec{v} = \langle a, b \rangle$ for $\lambda = 4$, I must solve the equation

$$(A - 4I)\vec{v} = \vec{0}, \quad \text{or} \quad \begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Solving this system amounts to row-reducing the matrix

$$\begin{bmatrix} 1 & -1 & 0 \\ 3 & -3 & 0 \end{bmatrix}.$$

However, row operations will not change the last column, so I'll omit it to save writing. I must remember to put it back when I write down the equations at the end.

$$\begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix} \underset{r_2 \to r_2 - 3r_1}{\longrightarrow} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

The last matrix gives the equation $a - b = 0$, so $a = b$. Therefore,

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} b \\ b \end{bmatrix} = b \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Any nonzero multiple of $\langle 1, 1 \rangle$ will be an eigenvector for $\lambda = 4$.

In what follows, I'll follow the procedure above without comment. That is, for each eigenvalue λ, I'll form the matrix $A - \lambda I$ and row-reduce it; remember that there is an extra column of zeros on the right that I'm not writing. Using the row-reduced form, I'll write down the corresponding system of equations — usually, I'll use a, b, c, ... as the variables. Finally, I'll write down an expression for the eigenvector(s) in terms of the free variables for the system.

Consider $\lambda = 2$. I follow the procedure I just described:

$$A - 2I = \begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix} \underset{r_2 \to r_2 - r_1}{\longrightarrow} \begin{bmatrix} 3 & -1 \\ 0 & 0 \end{bmatrix} \underset{r_1 \to r_1/3}{\longrightarrow} \begin{bmatrix} 1 & -\frac{1}{3} \\ 0 & 0 \end{bmatrix}$$

The last matrix says $a - \dfrac{1}{3}b = 0$, or $a = \dfrac{1}{3}b$. Therefore,

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \frac{1}{3}b \\ b \end{bmatrix} = b \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}.$$

Any nonzero multiple of $\left\langle \dfrac{1}{3}, 1 \right\rangle$ is an eigenvector for $\lambda = 2$. $\square$

16. Find all eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$$

The characteristic polynomial is $5 - 2\lambda + \lambda^2$. The eigenvalues are $\lambda = 1 \pm 2i$.
First, consider $\lambda = 1 + 2i$:

$$A - (1+2i)I = \begin{bmatrix} 2-2i & -2 \\ 4 & -2-2i \end{bmatrix} \underset{r_1 \to \frac{1}{2-2i} r_1}{\longrightarrow} \begin{bmatrix} 1 & -\frac{1}{2} - \frac{i}{2} \\ 4 & -2-2i \end{bmatrix} \underset{r_2 \to r_2 - 4r_1}{\longrightarrow} \begin{bmatrix} 1 & -\frac{1}{2} - \frac{i}{2} \\ 0 & 0 \end{bmatrix}$$

The last matrix says $a - \left(\frac{1}{2} + \frac{i}{2}\right) b = 0$, so $a = \left(\frac{1}{2} + \frac{i}{2}\right) b$. Therefore,

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2} + \frac{i}{2}\right) b \\ b \end{bmatrix} = b \begin{bmatrix} \frac{1}{2} + \frac{i}{2} \\ 1 \end{bmatrix}.$$

Consider $\lambda = 1 - 2i$:

$$A - (1-2i)I = \begin{bmatrix} 2+2i & -2 \\ 4 & -2+2i \end{bmatrix} \underset{r_1 \to \frac{1}{2+2i} r_1}{\longrightarrow} \begin{bmatrix} 1 & -\frac{1}{2} + \frac{i}{2} \\ 4 & -2+2i \end{bmatrix} \underset{r_2 \to r_2 - 4r_1}{\longrightarrow} \begin{bmatrix} 1 & -\frac{1}{2} + \frac{i}{2} \\ 0 & 0 \end{bmatrix}$$

The last matrix says $a - \left(\frac{1}{2} - \frac{i}{2}\right) b = 0$, so $a = \left(\frac{1}{2} - \frac{i}{2}\right) b$. Therefore,

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2} - \frac{i}{2}\right) b \\ b \end{bmatrix} = b \begin{bmatrix} \left(\frac{1}{2} - \frac{i}{2}\right) \\ 1 \end{bmatrix}. \square$$

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20. Find all eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & -4 \\ 4 & -7 \end{bmatrix}$$

The characteristic polynomial is $9 + 6\lambda + \lambda^2$. The eigenvalue is $\lambda = -3$ (double).
Consider $\lambda = -3$:

$$A + 3I = \begin{bmatrix} 4 & -4 \\ 4 & -4 \end{bmatrix} \underset{r_1 \to r_1 - r_2}{\longrightarrow} \begin{bmatrix} 4 & -4 \\ 0 & 0 \end{bmatrix} \underset{r_1 \to \frac{1}{4} r_1}{\longrightarrow} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

The last matrix says $a - b = 0$, so $a = b$. Therefore,

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} b \\ b \end{bmatrix} = b \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \square$$

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22. Find all eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 3 & 2 & 2 \\ 1 & 4 & 1 \\ -2 & -4 & -1 \end{bmatrix}$$

The characteristic polynomial is

$$6 - 11\lambda + 6\lambda^2 - \lambda^3 = (1 - \lambda)(2 - \lambda)(3 - \lambda).$$

The eigenvalues are $\lambda = 1$, $\lambda = 2$, and $\lambda = 3$.

Consider $\lambda = 3$:

$$A - 3I = \begin{bmatrix} 0 & 2 & 2 \\ 1 & 1 & 1 \\ -2 & -4 & -4 \end{bmatrix} \underset{r_1 \leftrightarrow r_2}{\rightarrow} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ -2 & -4 & -4 \end{bmatrix} \underset{r_3 \rightarrow r_3 + 2r_1}{\rightarrow}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & -2 & -2 \end{bmatrix} \underset{r_2 \rightarrow \frac{1}{2} r_2}{\rightarrow} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & -2 & -2 \end{bmatrix} \underset{r_3 \rightarrow r_3 + 2r_2}{\rightarrow}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \underset{r_1 \rightarrow r_1 - r_2}{\rightarrow} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The last matrix says $a = 0$ and $b + c = 0$, or $b = -c$. Therefore,

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ -c \\ c \end{bmatrix} = c \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}.$$

Next, consider $\lambda = 2$:

$$A - 2I = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 2 & 1 \\ -2 & -4 & -3 \end{bmatrix} \underset{r_2 \rightarrow r_2 - r_1}{\rightarrow} \begin{bmatrix} 1 & 2 & 2 \\ 0 & 0 & -1 \\ -2 & -4 & -3 \end{bmatrix} \underset{r_3 \rightarrow r_3 + 2r_1}{\rightarrow}$$

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} \underset{r_3 \rightarrow r_3 + r_2}{\rightarrow} \begin{bmatrix} 1 & 2 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \underset{r_2 \rightarrow -r_2}{\rightarrow}$$

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \underset{r_1 \rightarrow r_1 - 2r_2}{\rightarrow} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The last matrix says $c = 0$ and $a + 2b = 0$, or $a = -2b$. Therefore,

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -2b \\ b \\ 0 \end{bmatrix} = b \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}.$$

Finally, consider $\lambda = 1$:

$$A - I = \begin{bmatrix} 2 & 2 & 2 \\ 1 & 3 & 1 \\ -2 & -4 & -2 \end{bmatrix} \underset{r_1 \leftrightarrow r_2}{\rightarrow} \begin{bmatrix} 1 & 3 & 1 \\ 2 & 2 & 2 \\ -2 & -4 & -2 \end{bmatrix} \underset{r_2 \rightarrow r_2 - 2r_1}{\rightarrow}$$

$$\begin{bmatrix} 1 & 3 & 1 \\ 0 & -4 & 0 \\ -2 & -4 & -2 \end{bmatrix} \underset{r_3 \rightarrow r_3 + 2r_1}{\rightarrow} \begin{bmatrix} 1 & 3 & 1 \\ 0 & -4 & 0 \\ 0 & 2 & 0 \end{bmatrix} \underset{r_2 \rightarrow -\frac{1}{4} r_2}{\rightarrow}$$

$$\begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{bmatrix} \underset{r_3 \rightarrow r_3 - 2r_2}{\rightarrow} \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \underset{r_1 \rightarrow r_1 - 3r_2}{\rightarrow} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The last matrix says $b = 0$ and $a + c = 0$, or $a = -c$. Therefore,

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -c \\ 0 \\ c \end{bmatrix} = c \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}. \quad \square$$

"All men are ordinary men; the extraordinary men are those who know it." - G. K. Chesterton