Solutions to Problem Set 28.

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3. Find the general solution of the system of equations

$$\vec{x}' = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \vec{x}.$$

Also sketch a few of the trajectories and describe the behavior of the solutions as $t \to \infty$.

$$A = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}.$$

The characteristic polynomial is $-1 + \lambda^2$. The eigenvalues are $\lambda = \pm 1$.
Consider $\lambda = 1$:

$$A - I = \begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix} \underset{r_2 \to r_2 - 3r_1}{\to} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

The last matrix says $a - b = 0$, so $a = b$. Therefore,

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} b \\ b \end{bmatrix} = b \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Next, consider $\lambda = -1$:

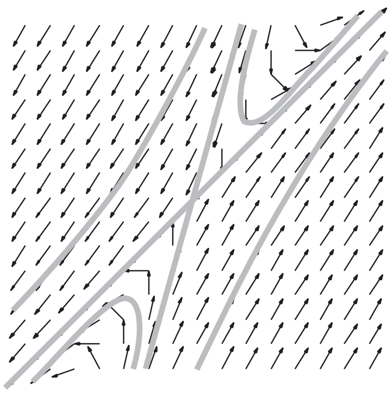
$$A + I = \begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix} \underset{r_2 \to r_2 - r_1}{\to} \begin{bmatrix} 3 & -1 \\ 0 & 0 \end{bmatrix} \underset{r_1 \to r_1/3}{\to} \begin{bmatrix} 1 & -\frac{1}{3} \\ 0 & 0 \end{bmatrix}$$

The last matrix says $a - \frac{1}{3}b = 0$, so $a = \frac{1}{3}b$. Therefore,

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \frac{1}{3}b \\ b \end{bmatrix} = b \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}.$$

The solution is

$$\vec{x} = c_1 e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}.$$



As $t \to \infty$, $x, y \to \pm\infty$. (They go to $+\infty$ if $c_1 > 0$ and $-\infty$ if $c_1 < 0$.) If $c_1 = 0$, $x, y \to 0$. □

8. Find the general solution of the system of equations

$$\vec{x}\,' = \begin{bmatrix} 2 & 10 \\ -1 & -5 \end{bmatrix} \vec{x}.$$

Also sketch a few of the trajectories. In each of these problems the coefficient matrix has a zero eigenvalue. As a result, the pattern of trajectories is different from those in the examples in the text.

$$A = \begin{bmatrix} 2 & 10 \\ -1 & -5 \end{bmatrix}.$$

The characteristic polynomial is $3\lambda + \lambda^2$. The eigenvalues are $\lambda = -3$ and $\lambda = 0$.
First, consider $\lambda = -3$:

$$A + 3I = \begin{bmatrix} 5 & 10 \\ -1 & -2 \end{bmatrix} \underset{r_1 \leftrightarrow r_2}{\rightarrow} \begin{bmatrix} -1 & -2 \\ 5 & 10 \end{bmatrix} \underset{r_2 \to r_2 + 5r_1}{\rightarrow} \begin{bmatrix} -1 & -2 \\ 0 & 0 \end{bmatrix} \underset{r_1 \to -r_1}{\rightarrow} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

The last matrix says $a + 2b = 0$, so $a = -2b$. Therefore,

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -2b \\ b \end{bmatrix} = b \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

Next, consider $\lambda = 0$:

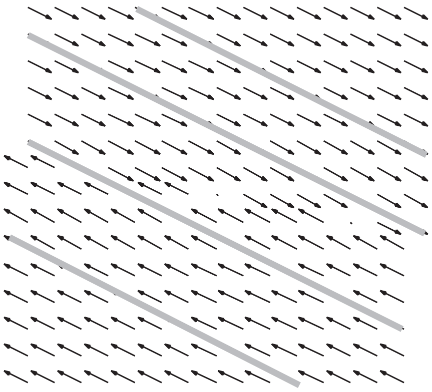
$$A = \begin{bmatrix} 2 & 10 \\ -1 & -5 \end{bmatrix} \underset{r_1 \leftrightarrow r_2}{\rightarrow} \begin{bmatrix} -1 & -5 \\ 2 & 10 \end{bmatrix} \underset{r_2 \to r_2 + 2r_1}{\rightarrow} \begin{bmatrix} -1 & -5 \\ 0 & 0 \end{bmatrix} \underset{r_1 \to -r_1}{\rightarrow} \begin{bmatrix} 1 & 5 \\ 0 & 0 \end{bmatrix}$$

The last matrix says $a + 5b = 0$, so $a = -5b$. Therefore,

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -5b \\ b \end{bmatrix} = b \begin{bmatrix} -5 \\ 1 \end{bmatrix}.$$

The solution is

$$\vec{x} = c_1 e^{-3t} \begin{bmatrix} -2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -5 \\ 1 \end{bmatrix}.$$

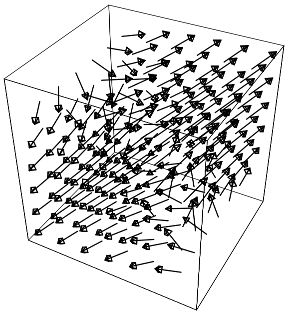


As $t \to \infty$, $\vec{x} \to c_2 \begin{bmatrix} -5 \\ 1 \end{bmatrix}$. Notice that the solutions are lines with slope $-\dfrac{1}{2}$. □

13. Find the general solution of the system of equations

$$\vec{x}' = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix} \vec{x}.$$

Here is the somewhat crowded vector field for the system:



$$A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}.$$

The characteristic polynomial is $8 + 15\lambda + 6\lambda^2 - \lambda^3$. The eigenvalues are $\lambda = 8$ and $\lambda = -1$ (double). Consider $\lambda = 8$:

$$A - 8I = \begin{bmatrix} -5 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & -5 \end{bmatrix} \underset{r_1 \leftrightarrow r_2}{\rightarrow} \begin{bmatrix} 2 & -8 & 2 \\ -5 & 2 & 4 \\ 4 & 2 & -5 \end{bmatrix} \underset{r_1 \to r_1/2}{\rightarrow}$$

$$\begin{bmatrix} 1 & -4 & 1 \\ -5 & 2 & 4 \\ 4 & 2 & -5 \end{bmatrix} \underset{r_2 \to r_2 + 5r_1}{\rightarrow} \begin{bmatrix} 1 & -4 & 1 \\ 0 & -18 & 9 \\ 4 & 2 & -5 \end{bmatrix} \underset{r_3 \to r_3 - 4r_1}{\rightarrow} \begin{bmatrix} 1 & -4 & 1 \\ 0 & -18 & 9 \\ 0 & 18 & -9 \end{bmatrix} \underset{r_3 \to r_3 + r_2}{\rightarrow}$$

$$\begin{bmatrix} 1 & -4 & 1 \\ 0 & -18 & 9 \\ 0 & 0 & 0 \end{bmatrix} \underset{r_2 \to -r_2/18}{\rightarrow} \begin{bmatrix} 1 & -4 & 1 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} \underset{r_1 \to r_1 + 4r_2}{\rightarrow} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

The last matrix says $a - c = 0$, so $a = c$, and $b - \frac{1}{2}c = 0$, so $b = \frac{1}{2}c$. Therefore,

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} c \\ \frac{1}{2}c \\ c \end{bmatrix} = c \begin{bmatrix} 1 \\ \frac{1}{2} \\ 1 \end{bmatrix}.$$

Next, consider $\lambda = -1$:

$$A + I = \begin{bmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{bmatrix} \underset{r_1 \leftrightarrow r_2}{\rightarrow} \begin{bmatrix} 2 & 1 & 2 \\ 4 & 2 & 4 \\ 4 & 2 & 4 \end{bmatrix} \underset{r_1 \to r_1/2}{\rightarrow} \begin{bmatrix} 1 & \frac{1}{2} & 1 \\ 4 & 2 & 4 \\ 4 & 2 & 4 \end{bmatrix}$$

$$\underset{r_2 \to r_2 - 4r_1}{\to} \begin{bmatrix} 1 & \frac{1}{2} & 1 \\ 0 & 0 & 0 \\ 4 & 2 & 4 \end{bmatrix} \underset{r_3 \to r_3 - 4r_1}{\to} \begin{bmatrix} 1 & \frac{1}{2} & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The last matrix says $a + \dfrac{1}{2}b + c = 0$, so $a = -\dfrac{1}{3}b - c$. Therefore,

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}b - c \\ b \\ c \end{bmatrix} = b \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

The solution is

$$\vec{x} = c_1 e^{8t} \begin{bmatrix} 1 \\ \frac{1}{2} \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + c_3 e^{-t} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}. \square$$

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17. Solve the initial value problem

$$\vec{x}' = \begin{bmatrix} -2 & 1 \\ -5 & 4 \end{bmatrix} \vec{x}, \quad \vec{v}(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

Describe the behavior of the solutions as $t \to \infty$.

$$A = \begin{bmatrix} -2 & 1 \\ -5 & 4 \end{bmatrix}.$$

The characteristic polynomial is $-3 - 2\lambda + \lambda^2$. The eigenvalues are $\lambda = -1$ and $\lambda = 3$. Consider $\lambda = -1$:

$$A + I = \begin{bmatrix} -1 & 1 \\ -5 & 5 \end{bmatrix} \underset{r_2 \to r_2 - 5r_1}{\to} \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \underset{r_1 \to -r_1}{\to} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

The last matrix says $a - b = 0$, so $a = b$. Therefore,

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} b \\ b \end{bmatrix} = b \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Next, consider $\lambda = 3$:

$$A - 3I = \begin{bmatrix} -5 & 1 \\ -5 & 1 \end{bmatrix} \underset{r_2 \to r_2 - r_1}{\to} \begin{bmatrix} -5 & 1 \\ 0 & 0 \end{bmatrix} \underset{r_1 \to -r_1/5}{\to} \begin{bmatrix} 1 & -\frac{1}{5} \\ 0 & 0 \end{bmatrix}$$

The last matrix says $a - \dfrac{1}{5}b = 0$, so $a = \dfrac{1}{5}b$. Therefore,

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \frac{1}{5}b \\ b \end{bmatrix} = b \begin{bmatrix} \frac{1}{5} \\ 1 \end{bmatrix}.$$

The solution is

$$\vec{x} = c_1 e^{t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} \frac{1}{5} \\ 1 \end{bmatrix}.$$

Now $\vec{x} = \langle 1, 3 \rangle$, so

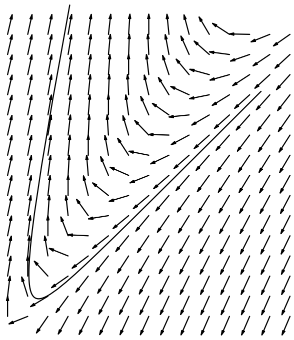
$$\begin{bmatrix} 1 \\ 3 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} \frac{1}{5} \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{5} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

Solve for c_1 and c_2:

$$\begin{bmatrix} 1 & \frac{1}{5} & 1 \\ 1 & 1 & 3 \end{bmatrix} \underset{r_2 \to r_2 - r_1}{\longrightarrow} \begin{bmatrix} 1 & \frac{1}{5} & 1 \\ 0 & \frac{4}{5} & 2 \end{bmatrix} \underset{r_2 \to 4r_2/5}{\longrightarrow} \begin{bmatrix} 1 & \frac{1}{5} & 1 \\ 0 & 1 & \frac{5}{2} \end{bmatrix} \underset{r_1 \to r_1 - r_2/5}{\longrightarrow} \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{5}{2} \end{bmatrix}$$

The last matrix says that $c_1 = \dfrac{1}{2}$ and $c_2 = \dfrac{5}{2}$. The solution is

$$\vec{x} = \frac{1}{2} e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{5}{2} e^{3t} \begin{bmatrix} \frac{1}{5} \\ 1 \end{bmatrix}.$$



As $t \to \infty$, $x, y \to +\infty$. $\square$

"I don't never have any trouble in regulating my own conduct, but to keep other folks' straight is what bothers me." - Josh Billings