Math 228
4-3-1994

Solutions to Problem Set 26.

[page 371]

1. Find the general solution of the system of equations

$$\vec{x}\,' = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} \vec{x}$$

The characteristic polynomial is $\lambda^2 - 2\lambda + 1 = (\lambda - 1)^2$. The eigenvalue is $\lambda = 1$ (double). Consider the eigenvalue $\lambda = 1$:

$$A - I = \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix} \quad \to \quad \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$$

The last matrix says that $a - 2b = 0$, so $a = 2b$. The eigenvector is

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2b \\ b \end{bmatrix} = b \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

The corresponding solution is

$$e^t \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

To obtain a second independent solution, "guess" a solution of the form $e^t\vec{u} + te^t\vec{v}$. Plug this into the equation $\vec{x}\,' = A\vec{x}$:

$$e^t\vec{u} + (te^t + e^t)\vec{v} = e^tA\vec{u} + te^tA\vec{v}, \quad (\vec{u} + \vec{v}) + t\vec{v} = A\vec{u} + tA\vec{v}.$$

Equate corresponding coefficients:

$$A\vec{v} = \vec{v}, \quad A\vec{u} = \vec{u} + \vec{v}.$$

Rewrite the equations:

$$(A - I)\vec{v} = \vec{0}, \quad (A - I)\vec{u} = \vec{v}.$$

The first equation says that $\vec{v}$ is an eigenvector for $\lambda = 1$, so take $\vec{v} = \langle 2, 1\rangle$. Write $\vec{u} = \langle u_1, u_2\rangle$, and plug $\vec{v}$ into the second equation:

$$\begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

I'll solve this equation by row reduction:

$$\begin{bmatrix} 2 & -4 & 2 \\ 1 & -2 & 1 \end{bmatrix} \quad \to \quad \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

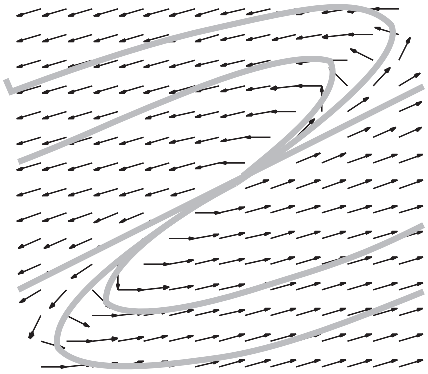
The last matrix says that $u_1 - 2u_2 = 1$, so $u_1 = 2u_2 + 1$. Set $u_2 = 1$: then $u_1 = 3$, so $\vec{u} = \langle 3, 1\rangle$. Therefore,

$$e^t\vec{u} + te^t\vec{v} = e^t \begin{bmatrix} 3 \\ 1 \end{bmatrix} + te^t \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

The general solution is

$$\vec{x} = c_1e^t \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \left(e^t \begin{bmatrix} 3 \\ 1 \end{bmatrix} + te^t \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right).$$

Here's a picture of the phase portrait:



It's an improper node. Notice the line $y = \dfrac{1}{2}x$ which corresponds to the eigenvector $\langle 2, 1\rangle$. $\square$

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6. Find the general solution of the system of equations

$$\vec{x}\,' = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \vec{x}$$

The characteristic polynomial is $-(\lambda + 1)^2(\lambda - 2)$; the eigenvalues are $\lambda = 2$ and $\lambda = -1$ (double). First, consider the eigenvalue $\lambda = 2$:

$$A - 2I = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

The last matrix says $a - c = 0$ and $b - c = 0$. Thus, $a = c$ and $b = c$, so the eigenvectors are

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} c \\ c \\ c \end{bmatrix} = c \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Next, consider the eigenvalue $\lambda = -1$:

$$A + I = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The last matrix says $a + b + c = 0$, or $a = -b - c$. Therefore, the eigenvectors are

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -b - c \\ b \\ c \end{bmatrix} = b \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

In this case, there are three independent eigenvectors, so the general solution is

$$\vec{x} = c_1 e^{2t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{-t} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Notice that just because you have a repeated root doesn't mean that you *won't* have enough independent eigenvectors. □

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7. Find the solution of the initial value problem

$$\vec{x}\,' = \begin{bmatrix} 1 & -4 \\ 4 & -7 \end{bmatrix} \vec{x}, \quad \vec{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

The characteristic polynomial is $\lambda^2 + 6\lambda + 9 = (\lambda + 3)^2$. The eigenvalue is $\lambda = -3$ (double). Consider the eigenvalue $\lambda = -3$:

$$A + 3I = \begin{bmatrix} 4 & -4 \\ 4 & -4 \end{bmatrix} \quad \to \quad \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

The last matrix says that $a - b = 0$, so $a = b$. The eigenvector is

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} b \\ b \end{bmatrix} = b \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The corresponding solution is

$$e^{-3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

To obtain a second independent solution, "guess" a solution of the form $e^t \vec{u} + te^t \vec{v}$.
Plug this into the equation $\vec{x}\,' = A\vec{x}$:

$$-3e^{-3t}\vec{u} + (-3te^{-3t} + e^{-3t})\vec{v} = e^{-3t}A\vec{u} + te^{-3t}A\vec{v}, \quad (-3\vec{u} + \vec{v}) - 3t\vec{v} = A\vec{u} + tA\vec{v}.$$

Equate corresponding coefficients:

$$A\vec{v} = -3\vec{v}, \quad A\vec{u} = -3\vec{u} + \vec{v}.$$

Rewrite the equations:

$$(A + 3I)\vec{v} = \vec{0}, \quad (A + 3I)\vec{u} = \vec{v}.$$

The first equation says that $\vec{v}$ is an eigenvector for $\lambda = -3$, so take $\vec{v} = \langle 1, 1 \rangle$.
Write $\vec{u} = \langle u_1, u_2 \rangle$, and plug $\vec{v}$ into the second equation:

$$\begin{bmatrix} 4 & -4 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

I'll solve this equation by row reduction:

$$\begin{bmatrix} 4 & -4 & 1 \\ 4 & -4 & 1 \end{bmatrix} \quad \to \quad \begin{bmatrix} 1 & -1 & \frac{1}{4} \\ 0 & 0 & 0 \end{bmatrix}$$

The last matrix says that $u_1 - u_2 = \dfrac{1}{4}$, so $u_1 = u_2 + \dfrac{1}{4}$. Set $u_2 = 0$: then $u_1 = \dfrac{1}{4}$, so $\vec{u} = \left\langle \dfrac{1}{4}, 0 \right\rangle$.
Therefore,

$$e^t \vec{u} + te^t \vec{v} = e^{-3t} \begin{bmatrix} \frac{1}{4} \\ 0 \end{bmatrix} + te^{-3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The general solution is

$$\vec{x} = c_1 e^{-3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \left(e^{-3t} \begin{bmatrix} \frac{1}{4} \\ 0 \end{bmatrix} + t e^{-3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right).$$

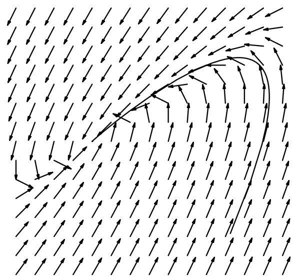
Finally, insert the initial condition:

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \vec{x}(0) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} \frac{1}{4} \\ 0 \end{bmatrix}.$$

This equation yields $c_1 = 2$, $c_2 = 4$. The solution is

$$\vec{x} = e^{-3t} \begin{bmatrix} 3 \\ 2 \end{bmatrix} + t e^{-3t} \begin{bmatrix} 4 \\ 4 \end{bmatrix}.$$

The picture below shows the direction field for the system, together with the particular solution for the problem:



This is an improper node; the eigenvector $\langle 1, 1 \rangle$ appears as the line $y = x$ in the picture. $\square$

"Characters that are depressed at a mere nothing are the kind best suited to endure heavy blows." - Cesare Pavese