Math 228
4-2-1994

Solutions to Problem Set 28.

1. The **Cayley-Hamilton theorem** says that a matrix A is annihilated by its characteristic polynomial $\det(A - \lambda I)$. Suppose

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}.$$

(a) Compute the characteristic polynomial of A.

The characteristic polynomial of A is $-5 - 4\lambda + \lambda^2$. □

(b) Show by direct computation that if you plug A into the characteristic polynomial, you obtain the zero matrix.

$$A^2 - 4A - 5I = \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix} + \begin{bmatrix} -4 & -16 \\ -8 & -12 \end{bmatrix} + \begin{bmatrix} -5 & 0 \\ 0 & -5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \square$$

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2. Let A be an n x n matrix, and let $\{\lambda_1, \ldots, \lambda_n\}$ be a list of the eigenvalues, with multiple eigenvalues repeated according to their multiplicity. The following recursive formula can be used to compute e^{tA}:

$$a_1 = e^{\lambda_1 t},$$

$$a_k = e^{\lambda_k t} \star a_{k-1}(t) = \int_0^t e^{\lambda_k (t-u)} a_{k-1}(u)\, du, \quad k = 2, \ldots, n,$$

$$B_1 = I,$$

$$B_k = (A - \lambda_{k-1} I) \cdot B_{k-1}, \quad k = 2, \ldots, n,$$

$$e^{tA} = a_1 B_1 + a_2 B_2 + \ldots + a_n B_n.$$

Use the recursive procedure above to solve the following linear systems:

(a) $\vec{x}\,' = \begin{bmatrix} 6 & 4 \\ -8 & -6 \end{bmatrix} \vec{x}$

The eigenvalues are $\lambda = 2$ and $\lambda = -2$. First, I'll compute the a_k's. $a_1 = e^{2t}$, and

$$a_2 = e^{-2t} \star e^{2t} = \int_0^t e^{-2(t-u)} e^{2u}\, du = e^{-2t} \int_0^t e^{-4u}\, du = \frac{1}{4} e^{2t} - \frac{1}{4} e^{-2t}.$$

Next, I'll find the B_k's:

$$B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B_2 = A - 2I = \begin{bmatrix} 4 & 4 \\ -8 & -8 \end{bmatrix}.$$

The exponential is

$$e^{tA} = a_1 B_1 + a_2 B_2 = \begin{bmatrix} -e^{-2t} + 2e^{2t} & -e^{-2t} + e^{2t} \\ 2e^{-2t} - 2e^{2t} & 2e^{-2t} - e^{2t} \end{bmatrix}.$$

The solution is

$$\vec{x} = \begin{bmatrix} -e^{-2t} + 2e^{2t} & -e^{-2t} + e^{2t} \\ 2e^{-2t} - 2e^{2t} & 2e^{-2t} - e^{2t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}. \square$$

(b) $\vec{x}\,' = \begin{bmatrix} -2 & -1 \\ 1 & -4 \end{bmatrix} \vec{x}$

Find the solution satisfying $\vec{x} = \langle 1, -3 \rangle$.

The eigenvalue is $\lambda = -3$ (double). $a_1 = e^{-3t}$, and

$$a_2 = e^{-3t} \star e^{-3t} = \int_0^t e^{-3(t-u)} e^{-3u}\, du = te^{-3t}.$$

Next,

$$B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B_2 = A + 3I = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}.$$

The exponential is

$$e^{tA} = a_1 B_1 + a_2 B_2 = \begin{bmatrix} e^{-3t} + te^{-3t} & -te^{-3t} \\ te^{-3t} & e^{-3t} - te^{-3t} \end{bmatrix}.$$

The solution satisfying the initial conditions is

$$\vec{x} = \begin{bmatrix} e^{-3t} + te^{-3t} & -te^{-3t} \\ te^{-3t} & e^{-3t} - te^{-3t} \end{bmatrix} \begin{bmatrix} 1 \\ -3 \end{bmatrix}. \square$$

(c) $\vec{x}\,' = \begin{bmatrix} -4 & -13 \\ 2 & 6 \end{bmatrix} \vec{x}$

The eigenvalues are $\lambda = 1 \pm i$. First,

$$a_1 = e^{(1+i)t} = e^t(\cos t + i \sin t).$$

The next computation is rather disgusting, and for that reason it might be better to use eigenvectors to solve the system when you have complex eigenvalues.

$$a_2 = e^{(1+i)t} \star e^{(1-i)t} = \int_0^t e^{(1+i)(t-u)} e^{(1-i)u}\, du = \int_0^t e^{(1+i)t} e^{(-1-i)u} e^{(1-i)u}\, du =$$

$$e^{(1+i)t} \int_0^t e^{-2iu}\, du = e^{(1+i)t} \left(\frac{-1}{2i} \right) \left[e^{-2iu} \right]_0^t = e^{(1+i)t} \left(\frac{i}{2} \right) \left(e^{-2it} - 1 \right) =$$

$$\frac{i}{2}\left(e^{(1-i)t}-e^{(1+i)t}\right)=\frac{i}{2}\left(e^t(\cos t-i\sin t)-e^t(\cos t+i\sin t)\right)=e^t\left(\frac{i}{2}\right)(-2i\sin t)=e^t\sin t.$$

Here are the B_k's:

$$B_1=\begin{bmatrix}1&0\\0&1\end{bmatrix},\quad B_2=A-(1+i)I=\begin{bmatrix}-5-i&-13\\2&5-i\end{bmatrix}.$$

The exponential is

$$e^{tA}=e^t\left(\begin{bmatrix}\cos t+i\sin t&0\\0&\cos t+i\sin t\end{bmatrix}+\begin{bmatrix}-5\sin t-i\sin t&-13\sin t\\2\sin t&5\sin t-i\sin t\end{bmatrix}\right)=$$

$$e^t\begin{bmatrix}\cos t-5\sin t&-13\sin t\\2\sin t&\cos t+5\sin t\end{bmatrix}.$$

The solution is

$$\vec{x}=e^t\begin{bmatrix}\cos t-5\sin t&-13\sin t\\2\sin t&\cos t+5\sin t\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix}.\ \square$$

(d) $\vec{x}\,'=\begin{bmatrix}1&-1&-1\\0&1&1\\0&0&1\end{bmatrix}\vec{x}$

The eigenvalue is $\lambda=1$ (triple). $a_1=e^t$, and

$$a_2=e^t\star e^t=\int_0^t e^{t-u}e^u\,du=te^t,$$

$$a_3=e^t\star te^t=\int_0^t e^{t-u}ue^u\,du=\frac{1}{2}t^2e^t.$$

Next,

$$B_1=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix},\quad B_2=A-I=\begin{bmatrix}0&-1&-1\\0&0&1\\0&0&0\end{bmatrix},\quad B_3=(A-I)(A-I)=\begin{bmatrix}0&0&-1\\0&0&0\\0&0&0\end{bmatrix}.$$

The exponential is

$$e^{tA}=\begin{bmatrix}e^t&-te^t&-te^t-\frac{1}{2}t^2e^t\\0&e^t&te^t\\0&0&e^t\end{bmatrix}.$$

The solution is

$$\vec{x}=\begin{bmatrix}e^t&-te^t&-te^t-\frac{1}{2}t^2e^t\\0&e^t&te^t\\0&0&e^t\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix}.\ \square$$

"We are not satisfied to be right, unless we can prove others to be quite wrong." - WILLIAM HAZLITT

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