Math 228
4-20-1994

Solutions to Problem Set 34.

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1. Verify that $(0,0)$ is a critical point, show that the system

$$\frac{dx}{dt} = x - y + xy,$$
$$\frac{dy}{dt} = 3x - 2y - xy,$$

is almost linear, and discuss the type and stability of the critical point $(0,0)$.

$(0,0)$ is a critical point, because when $x = 0$ and $y = 0$, $x - y + xy = 0$ and $3x - 2y - xy = 0$.
To show the system is almost linear, I write it in the form

$$\vec{x}\,' = A\vec{x} + \begin{bmatrix} \text{junk}_1 \\ \text{junk}_2 \end{bmatrix}.$$

Then I show that the components junk_1 and junk_2 are small near $\vec{0}$ — specifically,

$$\lim_{r\to 0} \frac{\text{junk}_1}{r} = 0 \quad \text{and} \quad \lim_{r\to 0} \frac{\text{junk}_2}{r} = 0.$$

(r is the polar coordinate r.)
Rewrite the system in the form

$$\vec{x}\,' = \begin{bmatrix} 1 & -1 \\ 3 & -2 \end{bmatrix} \vec{x} + \begin{bmatrix} xy \\ -xy \end{bmatrix}.$$

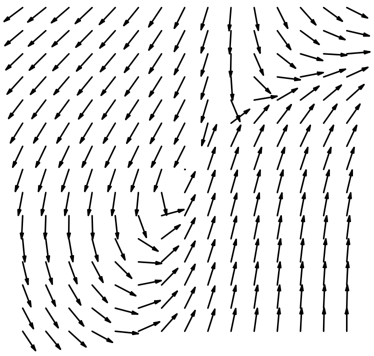
Then

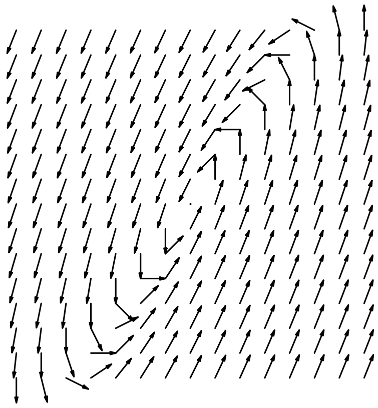
$$\lim_{r\to 0} \frac{xy}{r} = \lim_{r\to 0} \frac{r^2 \sin\theta\cos\theta}{r} = \lim_{r\to 0} r\sin\theta\cos\theta = 0.$$

Therefore, the system is almost linear.
Consider the corresponding linear system

$$\vec{x}\,' = \begin{bmatrix} 1 & -1 \\ 3 & -2 \end{bmatrix} \vec{x}.$$

The determinant is 1 and the trace is -1. Since $\det > \frac{1}{4}\text{tr}^2$, $(0,0)$ is a spiral; since $\text{tr} < 0$, it is asymptotically stable (a sink).





The picture on the left is the original system; the picture on the right is the corresponding linear system.

□

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4. Verify that $(0,0)$ is a critical point, show that the system

$$\frac{dx}{dt} = y + x(1 - x^2 - y^2),$$
$$\frac{dy}{dt} = -x + y(1 - x^2 - y^2),$$

is almost linear, and discuss the type and stability of the critical point $(0,0)$.

$(0,0)$ is a critical point, because when $x = 0$ and $y = 0$, $y + x(1-x^2-y^2) = 0$ and $-x + y(1-x^2-y^2) = 0$. Rewrite the system in the form

$$\vec{x}\,' = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \vec{x} + \begin{bmatrix} -x(x^2+y^2) \\ -y(x^2+y^2) \end{bmatrix}.$$

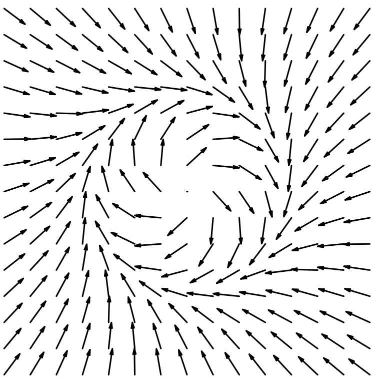
Then

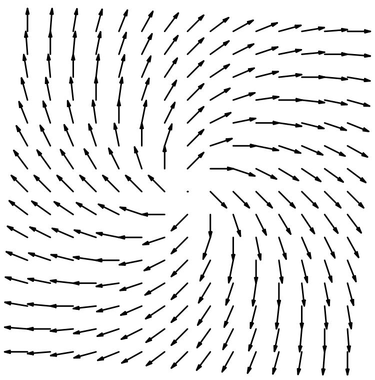
$$\lim_{r\to 0} \frac{-x(x^2+y^2)}{r} = \lim_{r\to 0} \frac{-r^3\cos\theta}{r} = \lim_{r\to 0} -r^2\cos\theta = 0,$$
$$\lim_{r\to 0} \frac{-y(x^2+y^2)}{r} = \lim_{r\to 0} \frac{-r^3\sin\theta}{r} = \lim_{r\to 0} -r^2\sin\theta = 0.$$

Therefore, the system is almost linear.
Consider the corresponding linear system

$$\vec{x}\,' = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \vec{x}.$$

The determinant is 2 and the trace is 2. Since $\det > \frac{1}{4}\,\mathrm{tr}^2$, $(0,0)$ is a spiral; since $\mathrm{tr} > 0$, it is unstable (a source).





The picture on the left is the original system; the picture on the right is the corresponding linear system.

□

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9. Verify that $(0,0)$ is a critical point, show that the system

$$\frac{dx}{dt} = (1+x)\sin y,$$
$$\frac{dy}{dt} = 1 - x - \cos y,$$

is almost linear, and discuss the type and stability of the critical point $(0,0)$.

$(0,0)$ is a critical point, because when $x=0$ and $y=0$, $(1+x)\sin y = 0$ and $1 - x - \cos y = 0$. First,

$$\frac{dx}{dt} = (1+x)\sin y = (1+x)\left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \cdots\right) = y + \left(xy - \frac{y^3}{3!} - x\frac{y^3}{3!} + \frac{y^5}{5!} + x\frac{y^5}{5!} - \cdots\right).$$

The linear part is y; converting to polar, I find that

$$\lim_{r\to 0}\frac{1}{r}\left(xy - \frac{y^3}{3!} - x\frac{y^3}{3!} + \frac{y^5}{5!} + x\frac{y^5}{5!} - \cdots\right) =$$

$$\lim_{r\to 0}\left(r\cos\theta\sin\theta - \frac{1}{3!}r^2(\sin\theta)^3 - \frac{1}{3!}r^3\cos\theta(\sin\theta)^3 + \cdots\right) = 0.$$

Likewise,

$$\frac{dy}{dt} = 1 - x - \cos y = 1 - x - 1 + \frac{y^2}{2!} - \frac{y^4}{4!} + \cdots = -x + \left(\frac{y^2}{2!} - \frac{y^4}{4!} + \cdots\right).$$

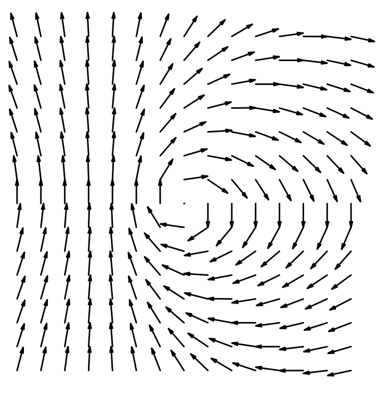
The linear part is $-x$; converting to polar, I find that

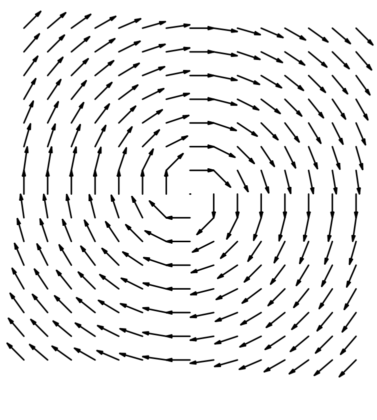
$$\lim_{r\to 0}\frac{1}{r}\left(\frac{y^2}{2!} - \frac{y^4}{4!} + \cdots\right) = \lim_{r\to 0}\left(\frac{1}{2!}r(\sin\theta)^2 - \frac{1}{4!}r^3(\sin\theta)^4 + \cdots\right) = 0.$$

Therefore, the system is almost linear.
Consider the corresponding linear system

$$\vec{x}\,' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\vec{x}.$$

The determinant is 1 and the trace is 0. The point $(0,0)$ is a stable center.





The picture on the left is the original system; the picture on the right is the corresponding linear system.

$\square$

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13. Determine all real critical points of the system of equations and discuss their type and stability:

$$\begin{aligned} \frac{dx}{dt} &= x - x^2 - xy, \\ \frac{dy}{dt} &= 3y - xy - 2y^2. \end{aligned}$$

$$\begin{array}{ccccc}
 & & x(1-x-y)=0 & & \\
 & \swarrow & & \searrow & \\
x=0 & & & 1-x-y=0 & \\
y(3-x-2y)=0 & & & y(3-x-2y)=0 & \\
\downarrow & \searrow & & y(2-y)=0 & \\
y=0 & & y=\dfrac{3}{2} & \downarrow & \searrow \\
(0,0) & & \left(0,\dfrac{3}{2}\right) & y=0 & y=2 \\
 & & & x=1 & x=-1 \\
 & & & (1,0) & (-1,2)
\end{array}$$

The critical points are $(0,0)$, $\left(0,\dfrac{3}{2}\right)$, $(1,0)$, and $(-1,2)$.

Compute the matrix of (first) partial derivatives:

$$\begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = \begin{bmatrix} 1-2x-y & -x \\ -y & 3-x-4y \end{bmatrix}.$$

If (a,b) is a critical point, then I can write

$$\frac{d}{dt}\begin{bmatrix} x-a \\ y-b \end{bmatrix} = \begin{bmatrix} 1-2a-b & -a \\ -b & 3-a-4b \end{bmatrix}\begin{bmatrix} x-a \\ y-b \end{bmatrix} + \begin{bmatrix} g_1(x,y) \\ g_2(x,y) \end{bmatrix}.$$

g_1 and g_2 are functions of quadratic order or higher near the critical point; the corresponding linear system is

$$\frac{d}{dt}\begin{bmatrix} x-a \\ y-b \end{bmatrix} = \begin{bmatrix} 1-2a-b & -a \\ -b & 3-a-4b \end{bmatrix}\begin{bmatrix} x-a \\ y-b \end{bmatrix}.$$

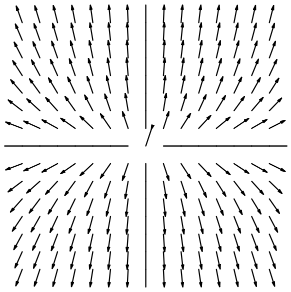
I just plug the critical points in one by one and classify the linear system. To save writing, I'll just write down the coefficient matrix

$$A = \begin{bmatrix} 1-2a-b & -a \\ -b & 3-a-4b \end{bmatrix}.$$

At $(0,0)$,

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}.$$

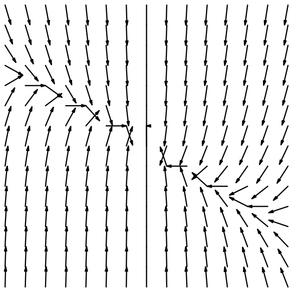
$\det = 3$, $\operatorname{tr} = 4$, so $\det < \dfrac{1}{4}\operatorname{tr}^2$. The point is an improper node; since $\operatorname{tr} > 0$, it's unstable. Here is the field for $-0.2 < x < 0.2$, $-0.2 < y < 0.2$:



At $\left(0, \frac{3}{2}\right)$,

$$A = \begin{bmatrix} -\frac{1}{2} & 0 \\ -\frac{3}{2} & -3 \end{bmatrix}.$$

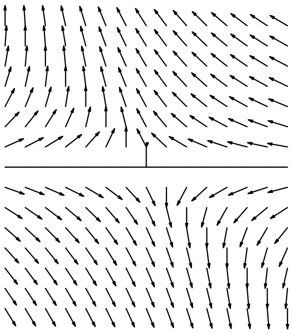
$\det = \frac{3}{2}$, $\mathrm{tr} = -\frac{7}{2}$, so $\det < \frac{1}{4}\mathrm{tr}^2$. The point is an improper node; since $\mathrm{tr} < 0$, it's asympotically stable. Here is the field for $-0.2 < x < 0.2$, $1.3 < y < 1.7$:



At $(1, 0)$,

$$A = \begin{bmatrix} -1 & -1 \\ 0 & 2 \end{bmatrix}.$$

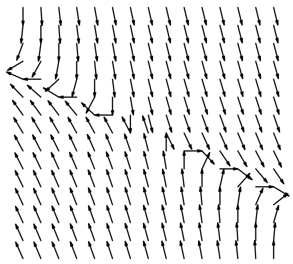
$\det = -2$ and $\mathrm{tr} = 1$. The point is an unstable saddle. Here is the field for $0.8 < x < 1.2$, $-0.2 < y < 0.2$:



At $(-1, 2)$,

$$A = \begin{bmatrix} 1 & 1 \\ -2 & -4 \end{bmatrix}.$$

det $= -2$ and tr $= -3$. The point is an unstable saddle. Here is the field for $-1.2 < x < -0.8$, $1.8 < y < 2.2$:



Notice that in each case, the magnified picture of the field near the critical point actually looks like the field type determined by the corresponding linear system! □

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19. In this problem we show how small changes in the coefficients of a system of linear equations can affect the nature of a critical point when the eigenvalues are equal. Consider the system

$$\vec{x}\,' = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \vec{x}.$$

Show that the eigenvalues are $r_1 = -1$, $r_2 = -1$ so that the critical points $(0, 0)$ is an asymptotically stable node. Now consider the system

$$\vec{x}\,' = \begin{bmatrix} -1 & 1 \\ -\epsilon & -1 \end{bmatrix} \vec{x}$$

where $|\epsilon|$ is arbitrarily small. Show that if $\epsilon > 0$, then the eigenvalues are $-1 \pm i\sqrt{\epsilon}$, so that the asymptotically stable node becomes an asymptotically stable spiral point. If $\epsilon < 0$, then the roots are $-1 \pm \sqrt{|\epsilon|}$, and the critical point remains an asymptotically stable node.

Consider the system

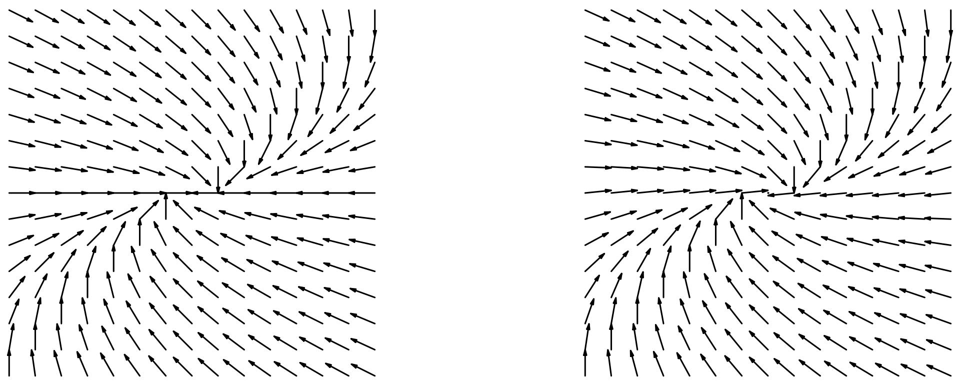
$$\vec{x}\,' = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \vec{x}.$$

Since the matrix is upper triangular, the eigenvalues are the diagonal entries: $r_1 = -1$, $r_2 = -1$. det $= 1$ and tr $= -2$, so det $= \frac{1}{4}\,\text{tr}^2$. The point $(0, 0)$ is a focus (or a node); since tr < 0, it is asymptotically stable.

Next, consider the system

$$\vec{x}\,' = \begin{bmatrix} -1 & 1 \\ -\epsilon & -1 \end{bmatrix} \vec{x}.$$

det $= 1 + \epsilon$, tr $= -2$, but now det $> \frac{1}{4}\,\text{tr}^2$. Hence, $(0, 0)$ is a spiral; since tr < 0, it is asymptotically stable.



The picture on the left is the original system (the node); the picture on the right is the perturbed system (the spiral). Even though the pictures are very similar, you *can* see one obvious difference: The node has a linear solution corresponding to its eigenvector, whereas the spiral has no linear solutions. □

"We should rather examine, who is better learned, than who is more learned." - MONTAIGNE