Math 228
4-24-1994

Solutions to Problem Set 35.

[page 471]

Each of Problems 1 through 6 can be interpreted as describing the interaction of two species with populations x and y. In each of these problems carry out the following steps.

(a) Find the critical points.

(b) For each critical point find the corresponding linear system. Find the eigenvalues and eigenvectors of the linear system; classify each critical point as to type, and determine whether it is asymptotically stable, stable, or unstable.

(c) Sketch the trajectories in the neighborhood of each critical point.

(d) If possible, compute and plot enough trajectories of the given system to show clearly the behavior of the solutions.

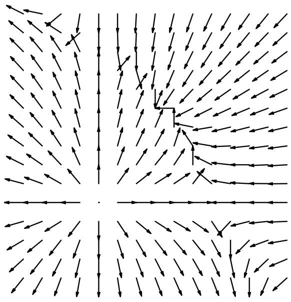
(e) Determine the limiting behavior of x and y as $t \to \infty$ and interpret the results in terms of the populations of the two species.

1.
$$\begin{aligned} \frac{dx}{dt} &= x(1.5 - x - 0.5y) \\ \frac{dy}{dt} &= y(2 - y - 0.75x) \end{aligned}$$

First, I'll find the critical points:

$$x(1.5 - x - 0.5y) = 0$$
$$\swarrow \qquad\qquad\qquad\qquad \searrow$$
$$\begin{array}{cccc} x = 0 & & y = 3 - 2x & \\ y(2 - y - 0.75x) = 0 & & y(2 - y - 0.75x) = 0 & \\ y(2 - y) = 0 & & (3 - 2x)(-1 + 1.25x) = 0 & \\ \downarrow & \searrow & \downarrow & \searrow \\ y = 0 & y = 2 & x = 3/2 & x = 4/5 \\ (0,0) & (0,2) & y = 0 & y = 7/5 \\ & & (3/2, 0) & (4/5, 7/5) \end{array}$$

The critical points are $(0,0)$, $(0,2)$, $\left(\frac{3}{2}, 0\right)$, and $\left(\frac{4}{5}, \frac{7}{5}\right)$.



If $x_0 \neq 0$ and $y_0 \neq 0$, $(x, y) \to \left(\frac{4}{5}, \frac{7}{5}\right)$ as $t \to \infty$. If $x_0 = 0$ and $y_0 \neq 0$, then $(x, y) \to (0, 2)$. If $x_0 \neq 0$ and $y_0 = 0$, then $(x, y) \to \left(\frac{3}{2}, 0\right)$. If $(x_0, y_0) = (0, 0)$, then $(x, y) = (0, 0)$ for all t.

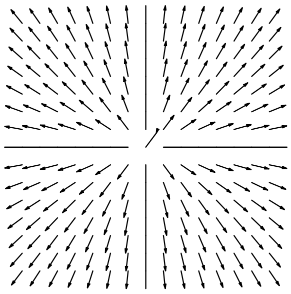
The matrix of partial derivatives is

$$A = \begin{bmatrix} 1.5 - 2x - 0.5y & -0.5x \\ -0.75y & 2 - 2y - 0.75x \end{bmatrix}.$$

At $(0, 0)$,

$$A = \begin{bmatrix} 1.5 & 0 \\ 0 & 2 \end{bmatrix}.$$

$\det = 3$, $\mathrm{tr} = 3.5$, so $\det < \frac{1}{4}\mathrm{tr}^2$. The point is an unstable improper node.

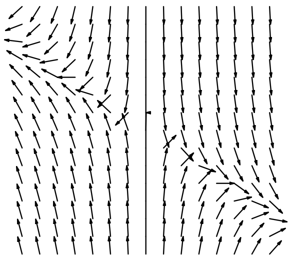


The eigenvalues are 1.5 and 2; the corresponding eigenvectors are $\langle 1, 0 \rangle$ and $\langle 0, 1 \rangle$.

At $(0, 2)$,

$$A = \begin{bmatrix} 0.5 & 0 \\ -1.5 & -2 \end{bmatrix}.$$

$\det = -1$, so the point is a saddle, and it's unstable.

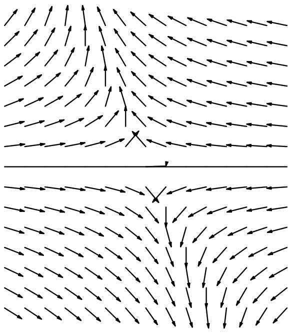


The eigenvalues are 0.5 and -2; the corresponding eigenvectors are $\langle 5, -3 \rangle$ and $\langle 0, 1 \rangle$.

At $\left(\frac{3}{2}, 0\right)$,

$$A = \begin{bmatrix} -1.5 & -0.75 \\ 0 & 0.875 \end{bmatrix}.$$

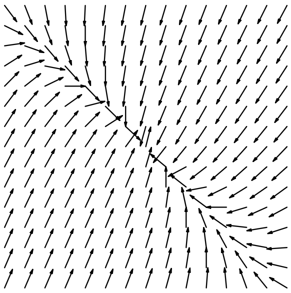
det $= -1.3125$, so the point is a saddle, and it's unstable.



The eigenvalues are 1.5 and -0.875; the corresponding eigenvectors are $\langle 1, 0\rangle$ and $\langle 6, -19\rangle$.
At $\left(\dfrac{4}{5}, \dfrac{7}{5}\right)$,

$$A = \begin{bmatrix} -0.8 & -0.4 \\ -1.05 & -1.4 \end{bmatrix}.$$

det $= 0.7$ and tr $= -2.2$, so det $< \dfrac{1}{4}\,\text{tr}^2$. The point is an asymptotically stable improper node.

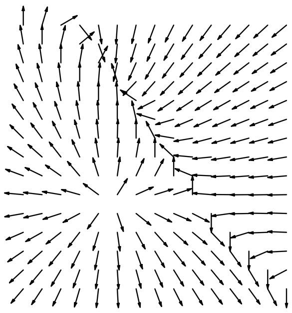


The eigenvalues are (approximately) -1.81414 and -38586; the corresponding eigenvectors are (approximately) $\langle 0.36691, 0.93026\rangle$ and $\langle 0.69472, -0.71928\rangle$. $\square$

5.
$$\frac{dx}{dt} = x(1 - x - y)$$
$$\frac{dy}{dt} = y(1.5 - y - x)$$

First, I'll find the critical points:

$$\begin{array}{ccccc}
 & & x(1-x-y) = 0 & & \\
 & \swarrow & & \searrow & \\
x = 0 & & & & y = 1 - x \\
y(1.5 - y - x) = 0 & & & & y(1.5 - y - x) = 0 \\
y(1.5 - y) = 0 & & & & (1 - x)(0.5) = 0 \\
\downarrow & \searrow & & & x = 1 \\
y = 0 & & y = 1.5 & & y = 0 \\
(0, 0) & & (0, 1.5) & & (1, 0)
\end{array}$$



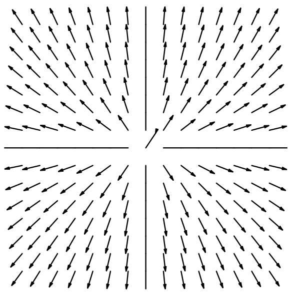
For $(x_0, y_0) \neq (0,0)$, $(x,y) \to (0, 1.5)$ as $t \to \infty$.
The matrix of partial derivatives is

$$A = \begin{bmatrix} 1 - 2x - y & -x \\ -y & 1.5 - 2y - x \end{bmatrix}.$$

At $(0,0)$,

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1.5 \end{bmatrix}.$$

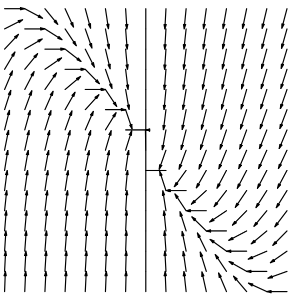
$\det = 1.5$, $\operatorname{tr} = 2.5$, so $\det < \dfrac{1}{4}\operatorname{tr}^2$. The point is an unstable improper node.



The eigenvalues are 1 and 1.5; the corresponding eigenvectors are $\langle 1, 0 \rangle$ and $\langle 0, 1 \rangle$.
At $(0, 1.5)$,

$$A = \begin{bmatrix} -0.5 & 0 \\ -1.5 & -1.5 \end{bmatrix}.$$

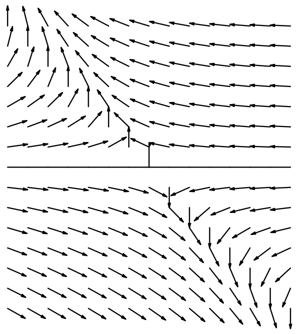
$\det = 0.75$, $\operatorname{tr} = -2$, so $\det < \dfrac{1}{4}\operatorname{tr}^2$. The point is an asymptotically stable improper node.



The eigenvalues are -0.5 and -1.5; the corresponding eigenvectors are $\langle 2,-3\rangle$ and $\langle 0,1\rangle$.
At $(1,0)$,

$$A = \begin{bmatrix} -1 & -1 \\ 0 & 0.5 \end{bmatrix}.$$

$\det = -0.5$, so the point is a saddle, and it's unstable.



The eigenvalues are -1 and 0.5; the corresponding eigenvectors are $\langle 1,0\rangle$ and $\langle 2,-3\rangle$. $\square$

"To achieve great things we must live as though we were never going to die." - VAUVENARGUES