Math 228
5-2-1994

Solutions to Problem Set 37.

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In each of Problems 1 through 4 construct a Liapunov function of the form $ax^2 + cy^2$, where a and c are to be determined. Then show that the critical point at the origin is of the indicated type.

For Problems 1 through 4, note once and for all that if $a, c > 0$, then $H(x,y) = ax^2 + cy^2$ satisfies $H(0,0) = 0$ and $H(x,y) > 0$ for $(x,y) \neq (0,0)$. Thus, H is positive definite.

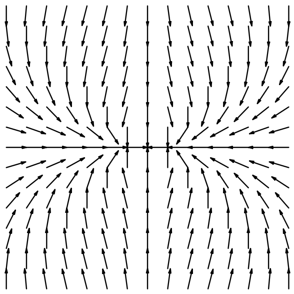
1. $$\frac{dx}{dt} = -x^3 + xy^2, \quad \frac{dy}{dt} = -2x^2y - y^3; \quad \text{asymptotically stable}$$

$$\nabla H \cdot \vec{x}\,' = \langle 2ax, 2cy\rangle \cdot \langle -x^3 + xy^2, -2x^2y - y^3\rangle = -2ax^4 + 2ax^2y^2 - 4cx^2y^2 - 2cy^4.$$

I will choose a and c so the middle terms cancel. Set $a = 2$, $c = 1$. Then

$$\nabla H \cdot \vec{x}\,' = -4x^4 - 2y^4 < 0 \quad \text{for} \quad (x,y) \neq (0,0).$$

It follows that $-\nabla H \cdot \vec{x}\,' > 0$ for $(x,y) \neq (0,0)$, so $-\nabla H \cdot \vec{x}\,'$ is positive definite. Hence, the origin is asymptotically stable.



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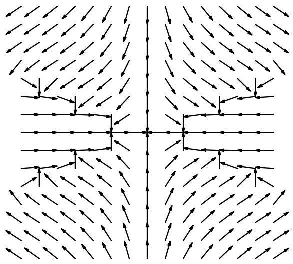
2. $$\frac{dx}{dt} = -\frac{1}{2}x^3 + 2xy^2, \quad \frac{dy}{dt} = -y^3; \quad \text{asymptotically stable}$$

$$\nabla H \cdot \vec{x}\,' = \langle 2ax, 2cy\rangle \cdot \left\langle -\frac{1}{2}x^3 + 2xy^2, -y^3\right\rangle = -ax^4 + 4ax^2y^2 - 2cy^4.$$

Set $a = 1$, $c = 8$. Then

$$\nabla H \cdot \vec{x}\,' = -x^4 + 4x^2y^2 - 16y^4 = -(x^2 - 2y^2)^2 - 12y^4 < 0 \quad \text{for} \quad (x,y) \neq (0,0).$$

It follows that $-\nabla H \cdot \vec{x}\,' > 0$ for $(x, y) \neq (0, 0)$, so $-\nabla H \cdot \vec{x}\,'$ is positive definite. Hence, the origin is asymptotically stable.



$\square$

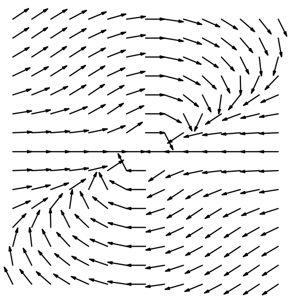
3.
$$\frac{dx}{dt} = -x^3 + 2y^3, \quad \frac{dy}{dt} = -2xy^2; \quad \text{stable (at least)}$$

$$\nabla H \cdot \vec{x}\,' = \langle 2ax, 2cy \rangle \cdot \langle -x^3 + 2y^3, -2xy^2 \rangle = -2ax^4 + 4axy^3 - 4cxy^3.$$

Set $a = c = 1$. The last two terms cancel, and

$$\nabla H \cdot \vec{x}\,' == -2x^4 \leq 0 \quad \text{for} \quad (x, y) \neq (0, 0).$$

(Notice that $-2x^4 = 0$ for points on the y-axis.) It follows that the origin is at least stable.



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5. Consider the system of equations

$$\frac{dx}{dt} = y - xf(x, y), \quad \frac{dy}{dt} = -x - yf(x, y),$$

where f is continuous and has continuous first partial derivatives. Show that if $f(x, y) > 0$ in some neighborhood of the origin, then the origin is an asymptotically stable critical point, and if $f(x, y) < 0$ in some neighborhood of the origin, then the origin is an unstable critical point.

$(0,0)$ is a critical point. If $c > 0$, then $H(x,y) = c(x^2 + y^2)$ satisfies $H(0,0) = 0$ and $H(x,y) > 0$ for $(x,y) \neq (0,0)$. Hence, H is positive definite.

Now

$$\nabla H \cdot \langle \vec{x}\,' = \langle 2cx, 2cy \rangle \cdot \langle y - xf, -x - yf \rangle = 2cxy - 2cx^2 f - 2cxy - 2cy^2 f = -2c(x^2 + y^2)f.$$

If $f(x,y) > 0$ in some neighborhood of the origin, then $-2c(x^2 + y^2)f < 0$ in some neighborhood of the origin. Therefore, the origin is an asymptotically stable critical point.

If $f(x,y) < 0$ in some neighborhood of the origin, then $-2c(x^2 + y^2)f > 0$ in some neighborhood of the origin. Hence, the origin is an unstable critical point. □

"Mankind are always happy for having been happy; so that if you make them happy now, you make them happy twenty years hence by the memory of it." - SYDNEY SMITH