Math 308
3-31-1994

# Problem Set 8.

Problem Set 8 is due Monday, April 11.

1. Let $V$ be an inner product space with inner product $f : V \times V \to F$, where $F = \mathbb{R}$ or $F = \mathbb{C}$.

(a) Let $T : V \to V$ be a linear transformation such that

$$f\left(T(x), T(y)\right) = f(x, y)$$

for all $x, y \in V$. Prove that $T$ is injective. Conclude that if $V$ is finite-dimensional, then $T$ is an isomorphism.

An isomorphism of vector spaces $T : V \to V$ is **unitary** if

$$f\left(T(x), T(y)\right) = f(x, y)$$

for all $x, y \in V$. ("**Unitary**" is usually used when $F = \mathbb{C}$; if $F = \mathbb{R}$, $T$ is **orthogonal**.)

(b) Prove that the set of unitary (orthogonal) transformations $V \to V$ forms a group under function composition.

The group of unitary transformations $\mathbb{C}^n \to \mathbb{C}^n$ relative to the standard inner product

$$\langle a_1, \ldots, a_n\rangle \cdot \langle b_1, \ldots, b_n\rangle = a_1\overline{b_1} + \cdots + a_n\overline{b_n}$$

is the **unitary group** $U_n(\mathbb{C})$.
The group of orthogonal transformations $\mathbb{R}^n \to \mathbb{R}^n$ relative to the standard inner product

$$\langle a_1, \ldots, a_n\rangle \cdot \langle b_1, \ldots, b_n\rangle = a_1 b_1 + \cdots + a_n b_n$$

is the **orthogonal group** $O_n(\mathbb{R})$.

(c) Show that an orthogonal transformation $T$ preserves lengths: $|Tx| = |x|$. Show that an orthogonal transformation preserves angles: The angle between $x$ and $y$ is the same (in magnitude) as the angle between $Tx$ and $Ty$.

A $n \times n$ real matrix is **orthogonal** if it is the matrix of an orthogonal transformation $\mathbb{R}^n \to \mathbb{R}^n$ relative to some basis on $\mathbb{R}^n$.

(d) Show that the rows of an orthogonal matrix form an orthonormal set. Show that the same is true of the columns.

That is, if $A$ is orthogonal and $A^T$ is the transpose of $A$, then $AA^T = I = A^T A$ — i.e. $A^T = A^{-1}$.

(e) Let $A$ be an orthogonal matrix. Prove that $\det A = \pm 1$.

(f) Show that the matrix of an orthogonal transformation $T \in O_2(\mathbb{R})$ has the form

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}.$$

Conclude that an orthogonal transformation of $\mathbb{R}^2$ is either a rotation or a reflection.

Let

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

be the matrix of $T \in O_2(\mathbb{R})$. Apply it to the vectors $\langle 1, 0\rangle$. Use the fact that $T$ preserves lengths to show that there is an angle $\theta$ such that $a = \cos\theta$, $c = \sin\theta$.

Now apply the matrix to $\langle 0, 1\rangle$ and $\langle 1, 1\rangle$. Again using the fact that lengths are preserved, solve for $b$ and $d$ in terms of $\theta$.

2. An $n \times n$ matrix $A$ is **upper triangular** if $A_{ij} = 0$ for $i > j$. (Thus, the elements below the main diagonal are 0.) Prove by induction that the determinant of an upper triangular matrix is the product of the entries on the main diagonal.

3. (a) Let $I$ be the $m \times m$ identity matrix, 0 the $m \times n$ zero matrix, $C$ an $n \times m$ matrix, and $D$ an $n \times n$ matrix. Use cofactor expansion to prove that

$$\begin{vmatrix} I & 0 \\ C & D \end{vmatrix} = |D|.$$

(b) Let $A$ be an $m \times m$ matrix, 0 the $m \times n$ zero matrix, $C$ an $n \times m$ matrix, and $D$ an $n \times n$ matrix. Fix $C$ and $D$, and define a function $T : M_m(F) \to F$ by

$$T(A) = \begin{vmatrix} A & 0 \\ C & D \end{vmatrix}.$$

Prove that $T$ is alternating and linear in each row.

(c) Let $A$ be an $m \times m$ matrix, 0 the $m \times n$ zero matrix, $C$ an $n \times m$ matrix, and $D$ an $n \times n$ matrix. Use (a) and (b) to prove that

$$\begin{vmatrix} A & 0 \\ C & D \end{vmatrix} = |A||D|.$$

(d) Let $A$ be an $m \times m$ matrix, $B$ an $m \times n$ matrix, 0 the $n \times m$ zero matrix, and $D$ an $n \times n$ matrix. Use (c) to prove that

$$\begin{vmatrix} A & B \\ 0 & D \end{vmatrix} = |A||D|.$$

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"Never try to say something remarkable. It is sure to be wrong." - Mark Rutherford