Math 308
4-10-1994

Problem Set 9.

This problem set is due Monday, April 18.

1. (a) Let A be an $m \times m$ matrix, B an $m \times n$ matrix, 0 the $n \times m$ zero matrix, and D an $n \times n$ matrix. Prove that the characteristic polynomial of

$$\begin{vmatrix} A & B \\ 0 & D \end{vmatrix}$$

is the product of the characteristic polynomials of A and D.

(b) Find *by hand* the characteristic polynomial and the eigenvalues of

$$\begin{bmatrix} 2 & -2 & 0 & 0 \\ 2 & -3 & 0 & 0 \\ 0 & 0 & 4 & -3 \\ 0 & 0 & 2 & -1 \end{bmatrix}.$$

2. A 3×3 matrix A has eigenvalues $\lambda = 1$ (double) and $\lambda = 3$. The corresponding eigenvectors are $\langle 1, 1, 1\rangle$, $\langle 1, 1, 0\rangle$, and $\langle 0, 1, 1\rangle$. Find A.

3. The **trace** of a matrix is the sum of the entries on the main diagonal:

$$\operatorname{tr} A = \sum_{i=1}^{n} A_{ii}, \qquad A \in M_n(F).$$

(a) Let $A \in M_n(F)$. Prove that

$$|\lambda I - A| = \lambda^n - \operatorname{tr} A \lambda^{n-1} + \cdots + (-1)^n \det A.$$

Hints: To get the constant term, set $\lambda = 0$. Be careful to explain the sign!

To prove that the trace gives the second term, induct on n. The result is easy for $n = 1, 2$. Take $n > 2$ and assume the result is true for $n - 1$. Let A be an $n \times n$ matrix. Expand $|\lambda I - A|$ by cofactors of the last row. What terms in the cofactor expansion can contribute terms λ^{n-1}?

(b) Use (a) to show that similar matrices have the same trace.

4. Find $A, B \in M_2(\mathbf{R})$ such that A and B have the same characteristic polynomial but A and B are not similar.

Hint: Remember that similar matrices have the same rank.

5. Find eigenvalues and eigenvectors for the following matrices over $\mathbb{C}$. You may use *Mathematica* to compute the eigenvalues and to solve systems of equations, but you may not use *Mathematica's* `Eigenvector` or `Eigensystem` commands (except to check your answers).

(a) $\begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & -3 \\ 0 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 3 & 1 & -1 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$

(Hint: If $a \pm bi$ is a conjugate pair of complex eigenvalues and $\vec{v}$ is an eigenvector for $a + bi$, then the complex conjugate $\vec{v}^*$ is an eigenvector for $a - bi$.)

"There is a courage of happiness as well as a courage of sorrow." - Maurice Maeterlinck